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Axisymmetric Stokes flow past a spherical cap

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The axisymmetric streaming Stokes flow past a body which contains a surface concave to the fluid is considered for the simplest geometry, namely, a spherical cap. It is found that a vortex ring is attached to the concave surface of the cap regardless of whether the oncoming flow is positive or negative. A stream surface $\psi = 0$ divides the vortex from the mainstream flow, and a detailed description of the flow is given for the hemispherical cup. The local velocity and stress in the vicinity of the rim are expressed in terms of local co-ordinates.

1. Introduction

The steady streaming Stokes flow of an incompressible viscous fluid past a fixed body is a problem which has been investigated by many authors during the past century. Stokes (see Lamb 1945) first gave the solution for an isolated sphere and later Oberbeck (1876) obtained the solution for the general ellipsoid. In 1960 Payne & Pell used the methods of generalized axially symmetric potential theory to calculate the flow past a class of axisymmetric bodies, including the lens, ellipsoid of revolution, spindle, and two separated spheres. More recently authors have been concerned with flow past slender bodies (see Cox 1970; Batchelor 1970). In most of these investigations the main result of physical interest is the drag formula for the particular obstacle under consideration. Apart from for the sphere, a detailed description of the streamlines has not been considered of prime interest even though explicit forms for the velocity field have been available. If the body is everywhere convex to the fluid it is fairly easy to visualize the streamlines. However, if the body is only partly convex and has a concave or re-entry region, e.g. the limaçon $r = 1+e\cos\theta$, $\frac{1}{2} < e < 1$, it is not clear whether the flow is separated by a stream surface $\psi = 0$ or whether the fluid particles flow in and out of such a region, so that upstream fluid particles eventually reach downstream infinity. To discuss this problem in general is a difficult task because even when explicit solutions are available for concave regions, e.g. the lens (Payne & Pell 1960), the velocity fields are of such a complicated functional form that considerable numerical work would be necessary to extract any useful information.

The present paper describes the streaming flow past the simplest fixed geometry possessing a concave region: the spherical cap (umbrella or parachute shape). The spherical cap is a particular case of the lens configuration in which both surfaces coincide and also contains the sphere and a thin circular disk as special limiting cases. Apart from possible applications in physiology (deformed red blood cells) and chemical engineering the cap has a mathematical interest because of its role in the theory of mixed boundary-value problems (Sneddon 1966). The problem of streaming Stokes flow was first discussed as a limiting form of the lens by Payne & Pell (1960) and later by Collins (1963) using a method of dual series equations. The method employed in this paper uses an integral transformation technique described in Ranger (1972) and Shail (1973) and it is possible to determine the stream function in a closed form, so that a fairly complete description of the flow is possible.

The velocity is continuous everywhere in the fluid region but the vorticity and pressure have singularities like the inverse square root of distance from the rim of the cap. In fact, in the solution presented here a unique solution is obtained by the requirement of zero velocity as the rim is approached. In Collins's solution uniqueness is achieved by minimizing the singularity in the pressure at the rim of the cap and the limiting value of the velocity is not considered. However, both methods lead to the same expression for the drag coefficient as a function of the cap angle. The advantages of the method of complementary integral representations employed here for the solution of the mixed boundary-value problem has been explained by Shail (1973) and it is worth pointing out that it is possible to discuss the flow in the neighbourhood of the rim without summing the series for the stream function. It would be necessary to sum the series for the stream function in the Collins paper to determine local expansions about the rim. However, it is routine procedure to determine the stream function in an elementary but cumbersome form and this has been carried out in the present paper so that streamlines can be plotted for the hemispherical cap.

If (r, θ) denote spherical co-ordinates and the cap is defined by $r = 1$, $0 \leqslant \theta \leqslant \alpha$, $0 < \alpha < \pi$, it is found that when $\alpha > 0$ a stream surface $\psi = 0$ forms in the fluid and is bounded by the rim of the cap. There is a stagnation ring inside this surface bounded by the cap, so that an axially symmetric vortex ring forms in which the fluid circulates about the stagnation ring. Since the flow is reversible the vortex will form in the fore or aft region of the flow depending on whether the concave or convex surface faces the oncoming stream. For all values of α, $0 < \alpha < \pi$, there is only one vortex inside the cap and there is no possibility of vortices with equal and opposite circulations occurring as the angle α approaches π. Detailed numerical calculations are given for the calotte or hemispherical cup $\alpha = \frac{1}{2}\pi$, and in particular it is found that the stagnation ring lies on the plane containing the rim of the cup. The direction in which the stream surface $\psi = 0$ leaves the rim is found for general values of α and the local velocity and stresses are calculated in a neighbourhood of the rim. It is found that in this vicinity the local flow, as expected, is two-dimensional and it appears that similar results can be found for a lens configuration containing a concave face, using the results of Moffatt (1964). However a detailed description of this flow is not considered here.

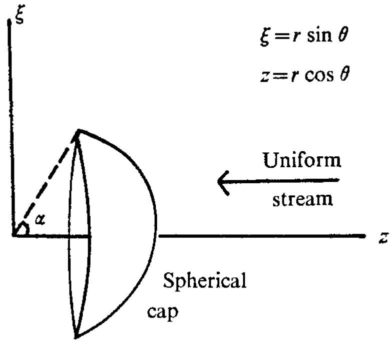

FIGURE 1

2. The method of complementary integral representations

The non-dimensional Stokes-flow equations for an incompressible viscous fluid are

$$\operatorname{grad} p = \nabla^2 \mathbf{q}, \quad \operatorname{div} \mathbf{q} = 0, \tag{2.1}$$

where $\mathbf{q}$ is the fluid velocity and p the fluid pressure. In axisymmetric flow the fluid velocity can be prescribed in terms of a stream function $\psi(r,\theta)$ by

$$\mathbf{q} = \operatorname{curl}\left\{-\frac{\psi(r,\theta)}{r\sin\theta}\hat{\boldsymbol{\phi}}\right\}, \tag{2.2}$$

where (r,θ) are spherical co-ordinates and $\hat{\boldsymbol{\phi}}$ is the unit vector perpendicular to the azimuthal plane $\phi =$ constant and in the sense of ϕ increasing. The radial and angular components of the fluid velocity are

$$q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \tag{2.3}$$

and $\psi(r,\theta)$ satisfies the repeated Stokes operator equation

$$L^2_{-1}(\psi) = 0, \quad L_{-1} \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{2.4}$$

The flow to be considered is the axisymmetric streaming motion past a fixed spherical cap described by $r = 1$, $0 \leqslant \theta \leqslant \alpha$ $(0 < \alpha \leqslant \pi$; see figure 1), and the boundary conditions are

$$\psi = \partial\psi/\partial r = 0, \quad r = 1, \quad 0 \leqslant \theta \leqslant \alpha, \tag{2.5}$$

$$\psi \sim \tfrac{1}{2}r^2\sin^2\theta, \quad r \to \infty. \tag{2.6}$$

The fluid velocity is to be continuous everywhere. In particular, continuity of $\mathbf{q}$ at the rim of the cap $(r = 1, \theta = \alpha)$ is required to provide a unique solution to the problem.

An appropriate representation for the stream function is

$$\psi = \tfrac{1}{2}r^2\sin^2\theta - V_1 + (r^2-1)\left(\tfrac{1}{2}r\,\partial V_1/\partial r - V_2\right), \tag{2.7}$$

where $V_1(r,\theta)$ and $V_2(r,\theta)$ both satisfy the Stokes equation $L_{-1}(V_j) = 0$, $j = 1,2$. Since

$$\left.\begin{aligned}\psi(1,\theta) &= \tfrac{1}{2}\sin^2\theta - V_1(1,\theta),\\ \partial\psi(1,\theta)/\partial r &= \sin^2\theta - 2V_2(1,\theta),\end{aligned}\right\} \tag{2.8}$$

the zero-velocity conditions become

$$V_j(1,\theta) = \tfrac{1}{2}\sin^2\theta, \quad 0 \leqslant \theta \leqslant \alpha. \tag{2.9}$$

As $r \to \infty$, $V_j = o(r^{-1})$.

A representation of the V_j in terms of complementary integrals is provided by

$$V_j(r,\theta) = \left\{\begin{aligned} r^{\frac{1}{2}}\int_0^\theta \frac{v_j(r,\lambda)\sin\lambda}{(\cos\lambda-\cos\theta)^{\frac{1}{2}}}d\lambda &= -r^{\frac{1}{2}}\int_\theta^\pi \frac{u_j(r,\lambda)\sin\lambda}{(\cos\theta-\cos\lambda)^{\frac{1}{2}}}d\lambda, \quad r \leqslant 1,\\ r^{\frac{1}{2}}\int_0^\theta \frac{v_j(r^{-1},\lambda)\sin\lambda}{(\cos\lambda-\cos\theta)^{\frac{1}{2}}}d\lambda &= -r^{\frac{1}{2}}\int_\theta^\pi \frac{u_j(r^{-1},\lambda)\sin\lambda}{(\cos\theta-\cos\lambda)^{\frac{1}{2}}}d\lambda, \quad r > 1.\end{aligned}\right\} \tag{2.10}$$

The functions (u_j, v_j), $j = 1, 2$, are conjugate two-dimensional harmonics even and odd in λ respectively and subject to the restrictions $v_j(r,0) = u_j(r,\pi) = 0$ for convergence. They are expressible in the form

$$u_j(r,\lambda) + iv_j(r,\lambda) = \frac{2^{\frac{3}{2}}}{\pi}\sum_{n=1}^{\infty} A_n^{(j)}(r\,e^{i\lambda})^{n+\frac{1}{2}}, \tag{2.11}$$

where the coefficients $A_n^{(j)}$ are real. Under the transforms given in (2.10), (u_j, v_j) map into

$$V_j(r,\theta) = \begin{cases}\displaystyle\sum_{n=1}^{\infty} A_n^{(j)} r^{n+1}\{P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\}, \quad r \leqslant 1, & (2.12\,a)\\ \displaystyle\sum_{n=1}^{\infty} A_n^{(j)}\, r^{-n}\{P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)\}, \quad r > 1, & (2.12\,b)\end{cases}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of degree n.

Setting $r = 1$ in (2.10), the boundary conditions on the cap (2.9) require

$$\tfrac{1}{2}\sin^2\theta = \int_0^\theta \frac{v_j(1,\lambda)\sin\lambda}{(\cos\lambda-\cos\theta)^{\frac{1}{2}}}d\lambda, \quad 0 \leqslant \theta \leqslant \alpha, \tag{2.13}$$

whose solution is

$$v_j(1,\lambda) = (2^{\frac{1}{2}}/3\pi)\sin\tfrac{3}{2}\lambda, \quad 0 \leqslant \lambda \leqslant \alpha. \tag{2.14}$$

On the part of the sphere $r = 1$ not occupied by the cap, the velocity and stress components are continuous and thus ψ, $\partial\psi/\partial r$, $\partial^2\psi/\partial r^2$ and $\partial^3\psi/\partial r^3$ are continuous on $r = 1$, $\alpha < \theta \leqslant \pi$. From (2.8) the first two conditions are satisfied because the V_j are continuous on $r = 1$. The third condition is satisfied if

$$\int_\theta^\pi \left(3\frac{\partial u_1}{\partial r} - 4\frac{\partial u_2}{\partial r}\right)\frac{\sin\lambda}{(\cos\theta-\cos\lambda)^{\frac{1}{2}}}d\lambda = 0, \quad r = 1, \quad \alpha < \theta \leqslant \pi, \tag{2.15}$$

or equivalently

$$3\frac{\partial u_1}{\partial r} - 4\frac{\partial u_2}{\partial r} = \frac{1}{r}\left(3\frac{\partial v_1}{\partial \lambda} - 4\frac{\partial v_2}{\partial \lambda}\right) = 0, \quad r = 1, \quad \alpha < \lambda \leqslant \pi. \tag{2.16}$$

The condition that $\partial^3\psi/\partial r^3$ be continuous reduces to

$$2\frac{\partial^3 v_1}{\partial\lambda^3}+\frac{1}{2}\frac{\partial v_1}{\partial\lambda}+2\frac{\partial v_2}{\partial\lambda}=0, \quad r=1, \quad \alpha<\lambda\leqslant\pi. \tag{2.17}$$

From (2.16), equation (2.17) can be expressed in terms of the derivatives of v_1 as

$$\partial^3 v_1/\partial\lambda^3+\partial v_1/\partial\lambda=0, \quad r=1, \quad \alpha<\lambda\leqslant\pi. \tag{2.18}$$

The general solutions for the $v_j(1,\lambda)$ are then of the form

$$\left.\begin{aligned} v_1(1,\lambda) &= C+D\cos\lambda+E\sin\lambda, \\ v_2(1,\lambda) &= B+\tfrac{3}{4}D\cos\lambda+\tfrac{3}{4}E\sin\lambda, \end{aligned}\right\} \quad \alpha<\lambda\leqslant\pi, \tag{2.19}$$

where B, C, D and E are constants to be determined. In order that q_r be finite as $\theta\to\pi, r=1$, it is necessary that $\partial v_1/\partial\lambda\to 0$ as $\lambda\to\pi, r=1$. This implies that $E=0$.

Expressions for $V_j(1,\theta)$ valid in the range $\alpha<\theta\leqslant\pi$ can now be obtained by setting $r=1$ in (2.10) and substituting for $v_j(1,\lambda)$ from (2.14) and (2.19). The results are

$$\begin{aligned} V_1(1,\theta) &= \int_0^\alpha \frac{v_1(1,\lambda)\sin\lambda}{(\cos\lambda-\cos\theta)^{\frac{1}{2}}}d\lambda+\int_\alpha^\theta \frac{v_1(1,\lambda)\sin\lambda}{(\cos\lambda-\cos\theta)^{\frac{1}{2}}}d\lambda \\ &= \frac{1}{\pi}\sin^2\theta\sin^{-1}\left\{\frac{\sin\frac{1}{2}\alpha}{\sin\frac{1}{2}\theta}\right\}+\left[\frac{2^{\frac{1}{2}}}{\pi}\sin\tfrac{1}{2}\alpha-\tfrac{4}{3}D\right](\cos\alpha-\cos\theta)^{\frac{3}{2}} \\ &\quad+\left[2C+2D\cos\alpha+\frac{2^{\frac{3}{2}}}{3\pi}(5\sin^3\tfrac{1}{2}\alpha-3\sin\tfrac{1}{2}\alpha)\right](\cos\alpha-\cos\theta)^{\frac{1}{2}}, \quad \alpha<\theta\leqslant\pi, \end{aligned} \tag{2.20}$$

$$\begin{aligned} V_2(1,\theta) &= \frac{1}{\pi}\sin^2\theta\sin^{-1}\left\{\frac{\sin\frac{1}{2}\alpha}{\sin\frac{1}{2}\theta}\right\}+\left[\frac{2^{\frac{1}{2}}}{\pi}\sin\tfrac{1}{2}\alpha-D\right](\cos\alpha-\cos\theta)^{\frac{3}{2}} \\ &\quad+\left[2B+\frac{3D}{2}\cos\alpha+\frac{2^{\frac{3}{2}}}{3\pi}(5\sin^3\tfrac{1}{2}\alpha-3\sin\tfrac{1}{2}\alpha)\right](\cos\alpha-\cos\theta)^{\frac{1}{2}}, \quad \alpha<\theta\leqslant\pi. \end{aligned} \tag{2.21}$$

Now from (2.12), $V_j(1,\pi)=0, j=1,2$. Also

$$q_r(1,\theta)=-\cos\theta+\frac{1}{\sin\theta}\frac{dV_1(1,\theta)}{d\theta}, \tag{2.22}$$

so in order that q_r be finite at the rim of the cap it is necessary that the singularity in $dV_1(1,\theta)/d\theta$ as $\theta\to\alpha+$ be eliminated. These conditions yield three equations for the determination of B, C and D which when solved give

$$\left.\begin{aligned} B &= (2^{\frac{1}{2}}/24\pi)(9\sin\tfrac{1}{2}\alpha+2\sin^3\tfrac{1}{2}\alpha), \\ C &= (2^{\frac{1}{2}}/6\pi)(3\sin\tfrac{1}{2}\alpha+2\sin^3\tfrac{1}{2}\alpha), \\ D &= (2^{\frac{1}{2}}/2\pi)(3\sin\tfrac{1}{2}\alpha). \end{aligned}\right\} \tag{2.23}$$

Thus from (2.14) and (2.19) the functions $v_j(1,\lambda)$ have been determined explicitly.

Taking the imaginary part of (2.11) and setting $r=1$ gives

$$v_j(1,\lambda)=\frac{2^{\frac{3}{2}}}{\pi}\sum_{n=1}^{\infty}A_n^{(j)}\sin(n+\tfrac{1}{2})\lambda. \tag{2.24}$$

But this is simply a Fourier series representation of $v_j(1,\lambda)$, so that the coefficients $A_n^{(j)}$ are given by

$$A_n^{(j)} = 2^{-\frac{1}{2}} \int_0^{\pi} v_j(1,\lambda) \sin(n+\tfrac{1}{2})\lambda \, d\lambda. \qquad (2.25)$$

Substituting for $v_j(1,\lambda)$ and integrating yields the coefficients explicitly:

$$A_n^{(1)} = \frac{1}{3\pi}\left[\frac{\sin(n-1)\alpha}{n-1} - \frac{\sin(n+2)\alpha}{n+2}\right] + \frac{2^{\frac{1}{2}}C}{2n+1}\cos(n+\tfrac{1}{2})\alpha + 2^{-\frac{1}{2}}D\left[\frac{\cos(n+\frac{3}{2})\alpha}{2n+3} + \frac{\cos(n-\frac{1}{2})\alpha}{2n-1}\right]; \qquad (2.26)$$

$A_n^{(2)}$ has the same form as $A_n^{(1)}$ with C replaced by B and D replaced by $\frac{3}{4}D$.

When $n = 1$, $\sin[(n-1)\alpha]/(n-1)$ is interpreted as α. These expressions for $A_n^{(j)}$ can now be substituted back into (2.12) to give infinite series expansions for the $V_j(r,\theta)$.

The drag on the cap can be computed from a formula due to Payne & Pell (1960). If U is the physical speed of the stream at infinity and a is the radius of the cap, the drag is

$$\begin{aligned} F &= 8\pi\rho\nu U a \lim_{r\to\infty} \frac{\psi - \frac{1}{2}r^2\sin^2\theta}{r\sin^2\theta} \\ &= -12\pi\rho\nu U a(\tfrac{1}{2}A_1^{(1)} + A_1^{(2)}) \\ &= -Ua\rho\nu(6\alpha + 8\sin\alpha + \sin 2\alpha), \end{aligned} \qquad (2.27)$$

where ρ and ν are the density and kinematic viscosity respectively. This expression agrees with Collins's calculation (1963) and predicts the correct drag on the sphere $\alpha = \pi$.

3. The flow near the rim of the cap

The solution given in the previous section is similar to that of Collins (1963) in that it is expressed in the form of an infinite series. The method of complementary integral representations used to derive the present solution, however, requires no knowledge of dual series equations and is easily adapted to problems involving the spherical cap where no axial symmetry exists (see Ranger 1973). In the next section it will be shown that the series in (2.12) can be summed explicitly to give the stream function in closed form, and although the same claim can be made for Collins's solution, the summation is more easily achieved in the present case. The real value of this method becomes evident however when one attempts to describe the flow near the rim of the cap. Because the cap is a concave body it is not clear whether the fluid flows around the rim into the concavity or whether a dividing stream surface $\psi = 0$ emanates from the rim and separates the fluid in the concavity from the external flow. In order to describe the flow completely therefore, it is necessary to understand the nature of the flow near the rim. Using Collins's solution such a 'rim analysis' would involve summing the series for the stream function explicitly and then expanding this function about the point $r = 1$, $\theta = \alpha$: a task which would involve enormous algebraic calculation. With

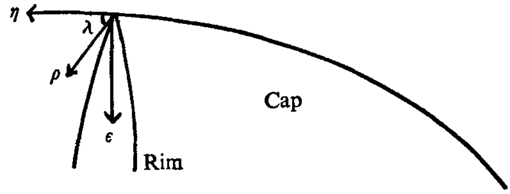

FIGURE 2

the present solution however, an asymptotic expression for the stream function near the rim is obtained without any algebraic hardship.

Define a local co-ordinate system (η, ϵ) at the rim in the following way:

$$r = 1-\epsilon, \quad \theta = \alpha+\eta. \tag{3.1}$$

Local polar co-ordinates are given by

$$\rho\, e^{i\lambda} = \eta + i\epsilon, \tag{3.2}$$

where $\rho \geqslant 0$, $-\pi \leqslant \lambda \leqslant \pi$. The inner surface of the cap is described by $\lambda = \pi$ and the outer surface by $\lambda = -\pi$ (figure 2).

In order to obtain an asymptotic expression for the stream function near the rim, asymptotic expressions for the V_j must be found. The relationship between ψ and the V_j in terms of local co-ordinates is, from (2.7),

$$\psi = \tfrac{1}{2}(1-\epsilon)^2 \sin^2(\alpha+\eta) - V_1 + (\epsilon - \tfrac{3}{2}\epsilon^2 + \tfrac{1}{2}\epsilon^3)\,\partial V_1/\partial\epsilon + (2\epsilon-\epsilon^2)\,V_2. \tag{3.3}$$

Now Moffatt (1964) has shown that, in a two-dimensional Stokes flow about a semi-infinite plate, the stream function near the edge of the plate will go as $\rho^{\frac{3}{2}}$, where ρ is distance from the edge. The situation in the present problem is analogous and so the stream function near the rim must go as $\rho^{\frac{3}{2}}$, where ρ is defined in (3.2). The leading term in the expansion for ψ is therefore denoted by $\psi_{\frac{3}{2}}$.

Since the ψ expansion contains half-integral powers of ρ, it follows from (3.3) that V_1 and V_2 must do likewise. Furthermore we see from (3.3) that the leading half-integral terms in the expansions of V_1 and V_2 are of order $\rho^{\frac{3}{2}}$ and $\rho^{\frac{1}{2}}$ respectively. Denote these terms by $V_{1,\frac{3}{2}}$ and $V_{2,\frac{1}{2}}$.

Recall that V_1 and V_2 satisfy the Stokes equation $L_{-1}(V_j) = 0$. An easy calculation reveals that, to first order, in the vicinity of the rim the Stokes operator is identical with the two-dimensional Laplacian operator. As a result the leading half-integral terms of V_1 and V_2 are harmonic functions of (ρ, λ). We may therefore write

$$V_{1,\frac{3}{2}}(\rho,\lambda) = f_1(\alpha)\,\rho^{\frac{3}{2}} \cos \tfrac{3}{2}\lambda + g_1(\alpha)\,\rho^{\frac{3}{2}} \sin \tfrac{3}{2}\lambda, \tag{3.4}$$

$$V_{2,\frac{1}{2}}(\rho,\lambda) = f_2(\alpha)\,\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\lambda + g_2(\alpha)\,\rho^{\frac{1}{2}} \sin \tfrac{1}{2}\lambda, \tag{3.5}$$

where $f_i(\alpha)$ and $g_i(\alpha)$ are coefficients to be determined.

Now let $\lambda = \pi$ in (3.4). In so doing we are evaluating $V_{1,\frac{3}{2}}$ along the inner cap surface. The result is

$$V_{1,\frac{3}{2}}(\rho,\pi) = -g_1(\alpha)\,\rho^{\frac{3}{2}}. \tag{3.6}$$

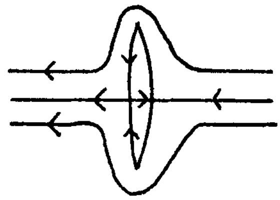

FIGURE 3

But $V_{1,\frac{3}{2}}(\rho,\pi)$ is the term of order $\rho^{\frac{3}{2}}$ in the expansion of $V_1(1,\theta)$ along the inner cap surface near the rim. In other words $V_{1,\frac{3}{2}}(\rho,\pi)$ is the $\rho^{\frac{3}{2}}$ term in $V_1(1,\alpha-\rho)$, which is known from (2.9). Clearly the expansion of

$$V_1(1,\alpha-\rho) = \tfrac{1}{2}\sin^2(\alpha-\rho) \tag{3.7}$$

contains no terms of order $\rho^{\frac{3}{2}}$, leading us to the conclusion that $g_1(\alpha) \equiv 0$.

Now set $\lambda = 0$ in (3.4). Using the same argument as above it follows that $f_1(\alpha)$ is the coefficient of $\rho^{\frac{3}{2}}$ in the expansion of $V_1(1,\alpha+\rho)$, which can be computed from (2.20). Similarly, from (2.9) and (2.21), $g_2(\alpha) \equiv 0$ and $f_2(\alpha)$ is the coefficient of $\rho^{\frac{1}{2}}$ in $V_2(1,\alpha+\rho)$. Carrying out the calculation it is found that

$$\left.\begin{aligned} f_1(\alpha) &= -(4/3\pi)(2\sin\alpha)^{\frac{1}{2}}\cos^3\tfrac{1}{2}\alpha, \\ f_2(\alpha) &= -(2\pi)^{-1}(2\sin\alpha)^{\frac{1}{2}}\sin\alpha\cos\tfrac{1}{2}\alpha. \end{aligned}\right\} \tag{3.8}$$

From (3.3), the relationship between $\psi_{\frac{3}{2}}$ and $(V_{1,\frac{3}{2}}, V_{2,\frac{1}{2}})$ is

$$\psi_{\frac{3}{2}} = -V_{1,\frac{3}{2}} + \epsilon\,\partial V_{1,\frac{3}{2}}/\partial\epsilon + 2\epsilon V_{2,\frac{1}{2}}, \tag{3.9}$$

where $\epsilon = \rho\sin\lambda$. Substituting (3.4) and (3.5) into (3.9) and simplifying, we obtain

$$\psi_{\frac{3}{2}}(\rho,\lambda) = 4\pi^{-1}(2\sin\alpha)^{\frac{1}{2}}\rho^{\frac{3}{2}}\cos^3\tfrac{1}{2}\alpha\cos^3\tfrac{1}{2}\lambda[\tfrac{1}{3}-\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\lambda]. \tag{3.10}$$

The most significant feature of (3.10) is that it predicts the existence of a dividing stream surface $\psi = 0$ for all cap angles $0 < \alpha < \pi$. This stream surface emanates from the rim of the cap and separates the external streaming flow ($\psi > 0$) from a sort of wake region within the cap's concavity ($\psi < 0$). The exact nature of this wake is discussed in §§ 5 and 6.

By setting $\psi_{\frac{3}{2}} = 0$ one can calculate the angle λ_0 at which the stream surface leaves the rim. The result is

$$\tan\tfrac{1}{2}\lambda_0 = \tfrac{1}{3}\cot\tfrac{1}{2}\alpha. \tag{3.11}$$

An interesting consequence of this formula is that $\lambda_0 \to \pi$ as $\alpha \to 0$. Thus when α is very small and the cap resembles a circular disk with slight spherical curvature, the wake is very thin and in fact vanishes in the limit (figure 3). This is consistent with Stokes flow past a circular disk, where no wake is present at all.

The velocity components near the rim are obtained from (3.10) using the formulae

$$q_\rho = \frac{-1}{\rho\sin\alpha}\frac{\partial\psi}{\partial\lambda}, \quad q_\lambda = \frac{1}{\sin\alpha}\frac{\partial\psi}{\partial\rho}. \tag{3.12}$$

The fluid velocity goes like $\rho^{\frac{1}{2}}$ and therefore is finite at the rim. However the pressure, vorticity and stress are given in term of velocity derivatives and these quantities have square-root singularities at the rim. This is in agreement with Collins's solution and is a consequence of the assumption that the cap has a sharp-edged rim with infinite curvature.

4. The stream function in closed form

In this section the functions $V_j(r,\theta)$, $j = 1, 2$, are obtained in closed form by summing the series in (2.12 *a*). This in turn permits the stream function to be written in closed form using (2.7). Because the interesting features of the flow lie within $r \leqslant 1$, attention is restricted to this region.

The first step in the procedure is to determine the generating function for the set of polynomials $\{P_{n-1}(x) - P_{n+1}(x)\}$, where $P_n(x)$ is the Legendre polynomial of degree n. This can be done using the identity

$$P_{n-1}(x) - P_{n+1}(x) \equiv -(2n+1)\int_{-1}^{x} P_n(x)\,dx, \tag{4.1}$$

together with the generating function for the Legendre polynomials

$$\sum_{n=0}^{\infty} P_n(x)\,t^n = (1-2xt+t^2)^{-\frac{1}{2}}, \quad |x| \leqslant 1, \quad |t| < 1. \tag{4.2}$$

Carrying out the necessary manipulations on (4.2) yields the required generating function:

$$\sum_{n=1}^{\infty} t^{n+1}\{P_{n-1}(x) - P_{n+1}(x)\} = 1 + xt - (1-2xt+t^2)^{\frac{1}{2}} - 2t(x-t)(1-2xt+t^2)^{-\frac{1}{2}}. \tag{4.3}$$

The series for $V_1(r,\theta)$ can be decomposed into five infinite series corresponding to the five terms in the expression for $A_n^{(1)}$ [see (2.26)]:

$$V_1(r,\theta) = \sum_{k=1}^{5} V_{1k}(r,\theta). \tag{4.4}$$

Each series $V_{1k}(r,\theta)$ is of the form

$$V_{1k}(r,\theta) = A\sum_{n=1}^{\infty}\left\{\frac{\sin(n+m)\alpha}{n+m} \quad \text{or} \quad \frac{\cos(n+m)\alpha}{n+m}\right\} r^{n+1}[P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)], \tag{4.5}$$

where A is some constant and m is some integer. The operations which convert (4.3) into a series of the form (4.5) are summarized by

$$t^{n+1} \rightarrow t^{n+m-1} \rightarrow \frac{t^{n+m}}{n+m} \rightarrow \frac{r^{n+m}\,e^{i\alpha(n+m)}}{n+m} \rightarrow \frac{r^{n+1}\,e^{i\alpha(n+m)}}{n+m}. \tag{4.6}$$

We begin by multiplying (4.3) through by t^{m-2} and then integrating with respect to t. Let $t = r\,e^{i\alpha}$ and then multiply through by r^{-m+1}. By choosing the real or imaginary part of the resulting expression we obtain a series of the form (4.5) together with its sum.

Each series $V_{1k}(r,\theta)$, $k = 1, \ldots, 5$, is summed in this manner and then the five of them are combined to give $V_1(r,\theta)$. The result is

$$\begin{aligned} V_1(r,\theta) = {} & \frac{\alpha}{2\pi} r^2 \sin^2\theta + \frac{1}{2\pi}\operatorname{Im}\{R(\tfrac{1}{3}e^{-2i\alpha} - \tfrac{1}{3}e^{i\alpha} + r\cos\theta\, e^{-i\alpha} - r^{-1}\cos\theta) \\ & + r^{-1}\sin^2\theta \ln[r\,e^{i\alpha} - \cos\theta + R] - r^2\sin^2\theta\ln[1 - r\cos\theta\, e^{i\alpha} + R]\} \\ & + 2^{\frac{1}{2}}C\operatorname{Re}\{e^{-\frac{1}{2}i\alpha}R\} + 2^{-\frac{1}{2}}D\operatorname{Re}\{\tfrac{1}{3}R(e^{\frac{1}{2}i\alpha} + e^{-\frac{3}{2}i\alpha}) \\ & + \tfrac{4}{3}\cos\theta\, e^{\frac{1}{2}i\alpha}R\,\frac{1 + r\,e^{-i\alpha}}{1 + r\,e^{i\alpha}} + \tfrac{2}{3}(r^{\frac{3}{2}} - r^{-\frac{1}{2}})(1-\cos\theta)\,F(s, \cos\tfrac{1}{2}\theta) \\ & + \tfrac{4}{3}(r^{\frac{3}{2}} - r^{-\frac{1}{2}})\cos\theta\, E(s, \cos\tfrac{1}{2}\theta)\}, \end{aligned} \tag{4.7}$$

where
$$R = (1-2r\cos\theta\, e^{i\alpha} + r^2 e^2 e^{2i\alpha})^{\frac{1}{2}},$$
$$s = \sin^{-1}\left[\frac{2r^{\frac{1}{2}}\, e^{\frac{1}{2}i\alpha}}{1+r\, e^{i\alpha}}\right],$$

$F(s, k)$ and $E(s, k)$ are incomplete elliptic integrals of the first and second kinds respectively, and C and D are constants defined in (2.23).

Although the calculation is not included here, the accuracy of (4.7) may be verified by showing that this expression satisfies boundary conditions (2.9) and (2.20). Since $A_n^{(2)}$ has the same form as $A_n^{(1)}$, $V_2(r,\theta)$ can be obtained directly from V_1 by replacing the constant C by B and D by $\frac{3}{4}D$. When the expressions for V_1 and V_2 are substituted into (2.7) and simplified, we obtain the stream function ψ in closed form:

$$\begin{aligned}\psi(r,\theta) = \frac{1}{2}\left(1-\frac{\alpha}{\pi}\right) r^2\sin^2\theta + \frac{1}{4\pi}\,\mathrm{Im}\Big\{&R(\tfrac{2}{3}r^2\, e^{i\alpha} - \tfrac{2}{3}r^2\, e^{-2i\alpha} - r^3\cos\theta\, e^{-i\alpha}\\
&+3r\cos\theta - r\ \cos\theta\, e^{-i\alpha} - r^{-1}\cos\theta) + (r^3-r)\,R'(\tfrac{1}{3}e^{-2i\alpha} - \tfrac{1}{3}e^{i\alpha}\\
&+r\cos\theta\, e^{-i\alpha} - r^{-1}\cos\theta) + (r^2-1)\sin^2\theta\,\frac{e^{i\alpha}+R'}{r\, e^{i\alpha}-\cos\theta+R}\\
&-(r^5-r^3)\sin^2\theta\,\frac{R'-\cos\theta\, e^{i\alpha}}{1+R-r\cos\theta\, e^{i\alpha}} + 2r^2\sin^2\theta\ln[1+R-r\cos\theta\, e^{i\alpha}]\\
&-(3r-r^{-1})\sin^2\theta\ln[r\, e^{i\alpha}-\cos\theta+R]\Big\}\\
&+2^{\frac{1}{2}}C\,\mathrm{Re}\{\tfrac{1}{2}(r^3-r)\, e^{-\frac{1}{2}i\alpha}\, R' - e^{-\frac{1}{2}i\alpha}\, R\} - 2^{\frac{1}{2}}B\,\mathrm{Re}\{(r^2-1)\, e^{-\frac{1}{2}i\alpha}\, R\}\\
&+2^{-\frac{1}{2}}D\,\mathrm{Re}\Big\{\tfrac{2}{3}(r^3-r)\cos\theta\,\frac{e^{-\frac{1}{2}i\alpha}-e^{-\frac{3}{2}i\alpha}}{(1+r\, e^{i\alpha})^2}\,R\\
&+(\tfrac{1}{6}r^3R' - \tfrac{1}{4}r^2R - \tfrac{1}{6}rR' - \tfrac{1}{12}R)\left(e^{\frac{1}{2}i\alpha} + e^{-\frac{3}{2}i\alpha} + 4\cos\theta\, e^{\frac{1}{2}i\alpha}\,\frac{1+r\, e^{-i\alpha}}{1+r\, e^{i\alpha}}\right)\\
&+\tfrac{1}{3}(r^2-1)^2(1-\ \cos\theta)\, e^{\frac{1}{2}i\alpha}\, R^{-1}\\
&+\tfrac{2}{3}(r^2-1)^2\cos\theta\, e^{\frac{1}{2}i\alpha}\, R(1+r\, e^{i\alpha})^{-2}\Big\},\end{aligned} \tag{4.8}$$

where $r \leqslant 1$ and
$$R' = (1-2r\cos\theta\, e^{i\alpha} + r^2\, e^{2i\alpha})^{\frac{1}{2}},$$
$$R' = \partial R/\partial r = (r\, e^{2i\alpha} - \cos\theta\, e^{i\alpha})\, R^{-1}.$$

Although this expression is rather long and cumbersome it consists of simple functions only. The incomplete elliptic integrals which were present in the expressions for the $V_j(r,\theta)$ cancel out and do not enter the formula for $\psi(r,\theta)$.

5. The flow within the wake

With the stream function expressed in closed form a more detailed description of the flow within the wake can be given. It is possible now to sketch the stream surfaces for any given α and this is done in the next section for the case of the hemispherical cup ($\alpha = \frac{1}{2}\pi$). In this section however, the magnitudes of the

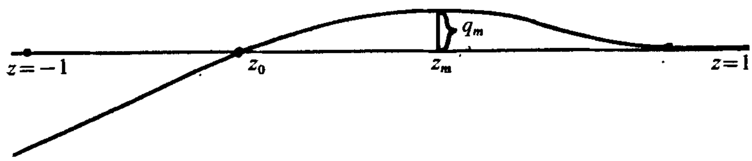

FIGURE 4

$12\pi^{-1}\alpha$	z_0	z_m	q_m
0	1·0	1·0	0·0
1	0·92	0·95	0·0017
2	0·72	0·82	0·0079
3	0·47	0·64	0·0160
4	0·18	0·44	0·0236
5	−0·10	0·22	0·0287
6	−0·35	0·0	0·0305
7	−0·57	−0·22	0·0287
8	−0·74	−0·42	0·0238
9	−0·86	−0·60	0·0167
10	−0·94	−0·76	0·0090
11	−0·99	−0·89	0·0030

TABLE 1

velocities within the wake are examined by considering the fluid velocity along the axis of symmetry.

The calculation of $q_z|_{\xi=0}$ from the stream function (4.8) is routine although lengthy. When $q_z|_{\xi=0}$ is plotted against z $(-1 \leqslant z \leqslant 1)$ we find that the curve has the same general shape regardless of the value of α (figure 4).

Since the flow at infinity is $q_z|_\infty = -1$, the region of positive fluid velocity $z_0 < z < 1$ verifies the existence of a wake within the cap's concavity. A stagnation point exists at $z = z_0$, where the dividing stream surface $\psi = 0$ intersects the axis. The axial velocity within the wake attains a maximum q_m at $z = z_m$. Of course z_0, z_m and q_m are functions of α and in table 1 and figure 5 the variation of these quantities with α is illustrated.

It is apparent that the fluid within the wake moves very slowly along the axis. Although exceedingly difficult to prove, it seems very likely that the curve of q_m *vs.* α has a maximum at $\alpha = \frac{1}{2}\pi$. As a result the axial fluid velocity within the wake never exceeds 3 % of the flow at infinity. This maximum velocity is attained at the origin when the cap is a hemispherical cup.

In figure 5 we see that z_0 and z_m have a cosine-like dependence on α. The curve $z = \cos\alpha$ gives the point where the plane determined by the rim of the cap intersects the axis. Thus the portion of the axis within the cap's concavity is described by $\cos\alpha < z < 1$. When $\alpha < \frac{1}{2}\pi$ both z_0 and z_m lie outside the concavity, but, as α increases, z_m is gradually overtaken by $\cos\alpha$. The hemispherical cup is unique in that its axial maximum occurs in the plane determined by the rim.

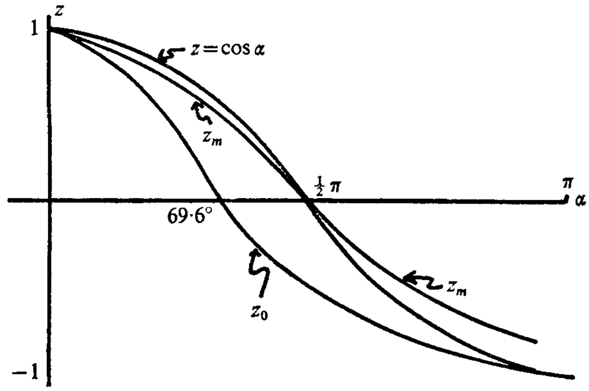

FIGURE 5

6. Stokes flow past a hemispherical cup

One of the major results in § 3 was that every spherical cap $0 < \alpha < \pi$ in Stokes flow exhibits a wake within its concavity. Then in § 5 we saw that the fluid velocity has the same general behaviour along the axis of symmetry regardless of the value of α. These two facts suggest that the nature of the wake is similar for all caps and that a representative picture of the flow can be obtained by sketching the stream surfaces for a particular case. Therefore in this section a sketch of the flow past a hemispherical cup ($\alpha = \frac{1}{2}\pi$) is made.

It is already known from (3.11) and table 1 that in the case of the cup the stream surface $\psi = 0$ leaves the rim at an angle of $\lambda_0 = 36{\cdot}9°$ and intersects the axis of symmetry at $z_0 = -0{\cdot}35$. The axial fluid velocity within the wake attains a maximum of $q_m = 5/3\pi - \frac{1}{2} = 0{\cdot}0305$ at $z_m = 0$. Further information is obtained when we calculate the fluid velocity in the plane $\theta = \frac{1}{2}\pi$ determined by the rim of the cap. This is rather tedious work involving expression (4.8) but when the calculation is finished a very interesting result emerges. We find that

$$q_r(r, \tfrac{1}{2}\pi) \equiv 0, \tag{6.1}$$

$$q_\theta(r, \tfrac{1}{2}\pi) = \frac{1}{2} - \frac{1}{4\pi}(3r^{-1} + r^{-3}) \sin^{-1} r - \frac{1}{4\pi}(3 - r^{-2})(1 - r^2)^{\frac{1}{2}}. \tag{6.2}$$

Equation (6.1) tells us that within the wake there is no flow along the plane $\theta = \frac{1}{2}\pi$. The fluid moves perpendicular to this plane. Furthermore the expression for $q_\theta(r, \frac{1}{2}\pi)$ vanishes when $r = r_0 = 0{\cdot}687$ and thus the flow has a stagnation ring at $r = 0{\cdot}687$, $\theta = \frac{1}{2}\pi$. Since expression (6.2) is negative when $r < r_0$ and positive when $r > r_0$, the fluid within the wake must execute a toroidal rotation about the stagnation ring. A sketch of the flow in the spherical region $r \leqslant 1$ is given in figure 6.

The flow past a hemispherical cup is representative of the Stokes flow past a spherical cap of any angle $0 < \alpha < \pi$. However it is not generally true that $q_r(r, \alpha) \equiv 0$ or that the stagnation ring lies in the plane $\theta = \alpha$ determined by the rim. In the case of the hemispherical cup, therefore, the task of determining the position of the stagnation ring is made easy by result (6.1). When $\alpha \neq \frac{1}{2}\pi$ the task is considerably more difficult.

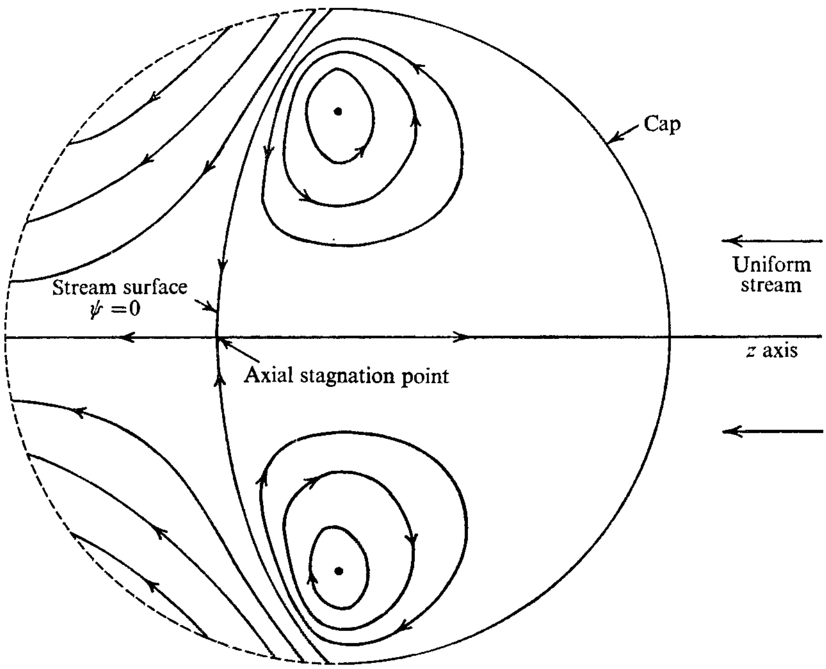

FIGURE 6

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Appendix

For the sake of completeness the stream function $\psi(r,\theta)$ is given in closed form for $r > 1$. It is apparent from (2.12 *b*) that, if $V_j(r,\theta) = W_j(r,\theta)$ when $r \leqslant 1$, then $V_j(r,\theta) = rW_j((r^{-1},\theta)$ when $r > 1$. Using (4.7) therefore, a closed-form expression for $V_1(r,\theta)$ when $r > 1$ is easily obtained and a similar one can be found for $V_2(r,\theta)$. When these two functions are combined in accordance with (2.7) the following expression results for $\psi(r,\theta)$, $r > 1$:

$$\psi(r,\theta) = \tfrac{1}{2}r^2\sin^2\theta - \frac{3\alpha}{4\pi}r\sin^2\theta + \frac{\alpha}{4\pi}\frac{\sin^2\theta}{r}$$

$$+\frac{1}{4\pi}\operatorname{Im}\Big\{R'(r^3-r)\left(\tfrac{1}{3}e^{-2i\alpha} - \tfrac{1}{3}e^{i\alpha} + r^{-1}\cos\theta\, e^{-i\alpha} - r\cos\theta\right)$$

$$+R\left(\tfrac{2}{3}r^2 e^{i\alpha} - \tfrac{2}{3}r^2 e^{-2i\alpha} + r^3\cos\theta - 3r\cos\theta\, e^{-i\alpha} + r\cos\theta\right.$$

$$+r^{-1}\cos\theta\, e^{-i\alpha} + (r^5-r^3)\sin^2\theta\,\frac{R'-\cos\theta}{e^{i\alpha}-r\cos\theta+R}$$

$$-(r^2-1)\sin^2\theta\,\frac{1+R'}{r-\cos\theta\, e^{i\alpha}+R} - 2r^2\sin^2\theta\ln\left[e^{i\alpha}-r\cos\theta+R\right]$$

[continued on next page]

$$+(3r-r^{-1})\sin^2\theta\ln[r-\cos\theta\, e^{i\alpha}+R]\Big\}+2^{\frac{1}{2}}C\,\mathrm{Re}\{\tfrac{1}{2}(r^3-r)$$

$$\times e^{-\frac{1}{2}i\alpha}R'-e^{-\frac{1}{2}i\alpha}R\}-2^{\frac{1}{2}}B\,\mathrm{Re}\{(r^2-1)\,e^{-\frac{1}{2}i\alpha}R\}$$

$$+2^{-\frac{1}{2}}D\,\mathrm{Re}\Big\{\tfrac{2}{3}(r^3-r)\cos\theta\,\frac{e^{\frac{3}{2}i\alpha}-e^{-\frac{1}{2}i\alpha}}{(r+e^{i\alpha})^2}R$$

$$+(\tfrac{1}{6}r^3R'-\tfrac{1}{4}r^2R-\tfrac{1}{6}rR'-\tfrac{1}{12}R)\left(e^{\frac{1}{2}i\alpha}+e^{-\frac{3}{2}i\alpha}+4\cos e^{\frac{1}{2}i\alpha}\frac{r+e^{-i\alpha}}{r+e^{i\alpha}}\right)$$

$$+\tfrac{1}{3}-(r^2-1)^2(1-\cos\theta)\,e^{\frac{1}{2}i\alpha}R^{-1}+\tfrac{2}{3}(r^2-1)^2\cos\theta\, e^{\frac{1}{2}i\alpha}R(r+e^{i\alpha})^{-2}\Big\},$$

where
$$R=(r^2-2r\cos\theta\, e^{i\alpha}+e^{2i\alpha})^{\frac{1}{2}},$$
$$R'=\partial R/\partial r=(r-\cos\theta\, e^{i\alpha})\,R^{-1}.$$

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